

Kondo problems in Tomonaga-Luttinger liquids

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Quantum impurity problems in Tomonaga-Luttinger liquids (TLLs) are reviewed with emphasis on their analogy to the Kondo problem in Fermi liquids. First, the problem of a static impurity in a spinless TLL is considered, which is related to the model studied in the context of the macroscopic quantum coherence. In the low-energy limit the TLL is essentially cut into two pieces when interaction is repulsive. The orthogonality catastrophe in a TLL is then discussed. Finally, the Kondo effect of a spin-1/2 impurity in a one-dimensional repulsively interacting electron liquids (a spinful TLL) is reviewed. Regardless of the sign of the exchange coupling, the impurity spin is completely screened in the ground state. The leading low-temperature contributions to thermodynamic quantities come from boundary contributions of a bulk leading irrelevant operator.

KEYWORDS: Kondo effect, Tomonaga-Luttinger liquid, bosonization, orthogonality catastrophe

1. Introduction

One-dimensional strongly interacting electrons have been a very actively studied subject for decades. In one spatial dimension mutual interactions among electrons have dramatic impact on low-energy properties of electron liquids and change them into Tomonaga-Luttinger liquids (TLLs) in which low-energy excitations are not fermionic single-particle excitations but bosonic gapless collective excitations of density fluctuations. The defining feature of a TLL is continuously varying exponents of correlation functions in its ground state, which experimentally manifests itself as power-law temperature dependence of response functions. For example, conductance of a junction formed in a TLL exhibits a power-law temperature dependence reflecting the energy dependence of tunnel density of states with an exponent depending on interaction strength. This characteristic feature is observed successfully in experiments of quantum wires,¹ edge states in fractional quantum Hall liquids,² and carbon nanotubes.³

Along with these experimental developments, quantum impurity problems in TLLs have been intensively studied theoretically. A minimal model among those studied is a spinless TLL with a static impurity potential.⁴ This model is closely related to a problem studied intensively in the context of macroscopic quantum coherence: quantum mechanics of a particle moving in a periodic potential with Ohmic dissipation. When the periodic potential is replaced by a double-well potential, the model is known to be equivalent to the Kondo problem.⁵ Moreover, at a particular value of a TLL parameter $g = \frac{1}{2}$, it is equivalent to the two-channel Kondo problem.⁶ An interesting generalization of these models is the one of a spinful TLL with a spin- $\frac{1}{2}$ magnetic impurity. A purpose of this paper is to review this Kondo problem in a TLL.⁷⁻¹⁰ In view of the above-mentioned similarity, we first review the static impurity problem (which will then serve as a basis for understanding the Kondo problem), the Anderson's orthogonality catastrophe in a TLL, and finally the Kondo problem. Since this paper is a short

article, we cannot cover many important developments such as exact results obtained through the Bethe ansatz. Instead, we will concentrate on low-energy effective field theory approach using the Abelian bosonization method.

2. A static impurity

2.1 spinless case

The starting point of our discussion is a simple continuum model of interacting spinless fermions with linear dispersion. The model can be applied to quantum wires in strong magnetic fields and to edge states in principal fractional quantum Hall liquids. Its Hamiltonian is written as

$$\mathcal{H}_0 = \int_{-\infty}^{\infty} dx \left\{ i v_F \left[\psi_L^\dagger(x) \partial_x \psi_L(x) - \psi_R^\dagger(x) \partial_x \psi_R(x) \right] + U \left[: \psi_L^\dagger(x) \psi_L(x) : + : \psi_R^\dagger(x) \psi_R(x) : \right]^2 \right\}, \quad (1)$$

where v_F is Fermi velocity, $\psi_L(x)$ and $\psi_R(x)$ are annihilation operators of left- and right-going fermions. We have discarded umklapp scattering, assuming that particle density is away from half filling. To simplify the Hamiltonian, we bosonize the fermions,

$$\psi_R(x) = \frac{e^{i\varphi_R(x)}}{\sqrt{2\pi\alpha}}, \quad \psi_L(x) = \frac{e^{-i\varphi_L(x)}}{\sqrt{2\pi\alpha}}, \quad (2)$$

where the bosonic fields obey the commutation relations

$$[\varphi_R(x), \varphi_R(y)] = -[\varphi_L(x), \varphi_L(y)] = i\pi \text{sgn}(x - y), \quad (3a)$$

$$[\varphi_R(x), \varphi_L(y)] = i\pi, \quad (3b)$$

and α is a short-distance cutoff. The particle density is written as

$$: \psi_R^\dagger(x) \psi_R(x) : = \frac{1}{2\pi} \frac{d\varphi_R}{dx}, \quad (4a)$$

$$: \psi_L^\dagger(x) \psi_L(x) : = \frac{1}{2\pi} \frac{d\varphi_L}{dx}, \quad (4b)$$

where the fermion operators are normal ordered with respect to the Dirac sea. The Hamiltonian is then

bosonized as

$$\mathcal{H}_0 = \frac{v}{8\pi} \int_{-\infty}^{\infty} dx \left[\frac{1}{g} \left(\frac{d\phi}{dx} \right)^2 + g \left(\frac{d\theta}{dx} \right)^2 \right], \quad (5)$$

where $g = [1 + (2U/\pi v_F)]^{-1/2}$, $v = v_F/g$, $\phi = \varphi_L + \varphi_R$, and $\theta = \varphi_L - \varphi_R$. The field ϕ represents particle density while the field θ corresponds to Josephson phase. The TLL parameter g is equal to 1 for the noninteracting case and smaller than 1 for repulsive interactions. The Hamiltonian (5) is the Gaussian model describing free massless bosons with linear dispersion $\omega = v|k|$.

Let us introduce a short-range impurity potential at $x = 0$ and discuss tunneling of particles through the potential barrier. The scattering by the impurity potential is described by

$$\begin{aligned} \lambda_F : \psi_L^\dagger(0) \psi_L(0) + (L \rightarrow R) : + \lambda_B [\psi_L^\dagger(0) \psi_R(0) + \text{H.c.}] \\ = \frac{\lambda_F}{2\pi} \frac{d\phi(0)}{dx} - \frac{\lambda_B}{\pi\alpha} \sin \phi(0). \end{aligned} \quad (6)$$

The coupling constants λ_F and λ_B characterize strength of forward and backward scattering by the impurity, respectively. The forward scattering can be absorbed by the transformation $\psi_L \rightarrow \psi_L e^{i(\lambda_F/v_F)\Theta(x)}$ and $\psi_R \rightarrow \psi_R e^{-i(\lambda_F/v_F)\Theta(x)}$, and thus we will not consider it further in this section. The backward scattering becomes a local nonlinear operator of the ϕ field and is renormalized by interactions.

Since the perturbation is localized at $x = 0$, we may integrate out the fields away from $x = 0$ and derive an effective action for the field $\phi_0 = \phi(x = 0)$. To this end, we write the partition function of the Gaussian model as

$$Z_0[\phi_0] = \int \mathcal{D}\phi(x, \tau) \mathcal{D}\lambda(\tau) e^{-S_0}, \quad (7)$$

where λ is an auxiliary field, and the action is

$$\begin{aligned} S_0 = \frac{v}{8\pi g} \int_0^{1/T} d\tau \int_{-\infty}^{\infty} dx \left[\left(\frac{\partial\phi}{\partial x} \right)^2 + \frac{1}{v^2} \left(\frac{\partial\phi}{\partial \tau} \right)^2 \right] \\ + i \int_0^{1/T} d\tau \lambda(\tau) [\phi_0(\tau) - \phi(0, \tau)]. \end{aligned} \quad (8)$$

We integrate out ϕ first and then λ to obtain the effective action for ϕ_0 ,

$$S_{\text{eff}} = \sum_{\omega_n} \frac{|\omega_n|}{4\pi g} |\tilde{\phi}_0(i\omega_n)|^2 - \frac{\lambda_B}{\pi\alpha} \int_0^{1/T} d\tau \sin \phi_0(\tau), \quad (9)$$

where $\omega_n = 2\pi T n$ ($n \in \mathbb{Z}$) and

$$\phi_0(\tau) = \sqrt{T} \sum_{\omega_n} e^{-i\omega_n \tau} \tilde{\phi}_0(i\omega_n). \quad (10)$$

The first term in eq. (9) is reminiscent of the Ohmic dissipation term in the Caldeira-Leggett action,⁵ which is caused by the gapless excitations in the TLL. If we added a “kinetic energy” term $m(\partial_\tau \phi_0)^2$ to regularize high-energy divergence, then our effective action would become the same as that used in the problem of macroscopic quantum coherence (MQC).

When $\lambda_B = 0$, it is easy to calculate two-time correlation function with the effective action, yielding

$$\langle e^{i\mu\phi_0(\tau)} e^{-i\mu\phi_0(0)} \rangle \propto \tau^{-2g\mu^2} \quad \text{at } T = 0. \quad (11)$$

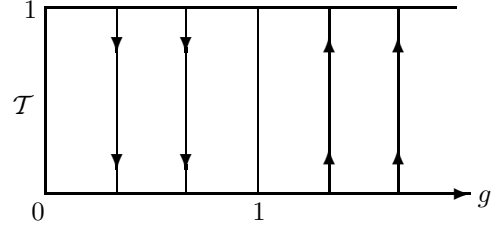


Fig. 1. Schematic flow diagram for the transmission probability. There is now flow at $g = 1$.

Equation (11) tells that the scaling dimension of the backward-scattering operator $\sin \phi_0$ is g at the Gaussian fixed point. Hence, in lowest order, the scaling equation for λ_B is given by

$$\frac{d\lambda_B}{dl} = (1 - g)\lambda_B, \quad (12)$$

where $dl = -d\Lambda/\Lambda$ with Λ high-energy cutoff. The TLL parameter g , on the other hand, is not renormalized since the local perturbation cannot affect the coupling constant in the bulk fixed-point Hamiltonian \mathcal{H}_0 . The scaling equation (12) indicates that, as energy scale decreases, the backscattering becomes stronger (weaker) for $g < 1$ ($g > 1$). At the noninteracting point $g = 1$, the coupling λ_B is not renormalized.

When the coupling λ_B is large, we cannot rely on the perturbative scaling equation (12). In this case the field ϕ_0 is almost always pinned at $\phi_0 = \pi/2 \pmod{2\pi}$ and occasionally changes by $\pm 2\pi$. To examine if a tunneling event, say, from $\phi_0 = \pi/2$ to $\phi_0 = 5\pi/2$ at time τ_0 is relevant or not, we substitute $\phi_0(\tau) = \pi/2 + 2\pi\Theta(\tau - \tau_0)$ into the action (9),

$$S_{\text{eff}} + \frac{\lambda_B}{\pi\alpha T} = \frac{2\pi T}{g} \sum_{\omega_n} \frac{1 - \cos(\omega_n \tau_0)}{|\omega_n|} \propto \frac{2}{g} \ln \tau_0. \quad (13)$$

We can conclude from eq. (13) that the scaling equation for fugacity t of the tunneling is

$$\frac{dt}{dl} = \left(1 - \frac{1}{g}\right) t. \quad (14)$$

The tunneling is relevant (irrelevant) for $g > 1$ ($g < 1$). Note the duality between eqs. (12) and (14).

Let us denote by \mathcal{T} transmission probability of a density wave through the potential barrier. Combining the scaling equations (12) and (14), we can draw a schematic renormalization-group flow diagram⁴ (Fig. 1). For repulsive interactions ($g < 1$) the transmission probability is renormalized down to zero, and a density wave is perfectly reflected in the low-energy limit. The TLL is effectively cut into two pieces by an impenetrable barrier. On the contrary, for attractive interactions ($g > 1$) the density wave is free to transmit through the barrier in the low-energy limit, in agreement with quasi-long-range order of superconductivity. At the noninteracting point $g = 1$, the potential is marginal and not renormalized at all, in agreement with our common knowledge that the transmission probability of a noninteracting particle can take any value (between 0 and 1) and is a smooth function of the potential strength.

In the MQC problem it is well known that an explicit solution of the problem is possible at a particular value of the dissipation strength which corresponds to $g = 1/2$ in our problem. At this point the scaling dimension of $\sin \phi_0$ becomes $1/2$, indicating that this operator can be written as a fermion operator. To see this, let us introduce new chiral bosonic fields¹¹

$$\varphi_{\pm}(x) = \frac{1}{\sqrt{8}} \left\{ \left(\frac{1}{\sqrt{g}} - \sqrt{g} \right) [\varphi_R(x) \pm \varphi_L(-x)] + \left(\frac{1}{\sqrt{g}} + \sqrt{g} \right) [\varphi_L(x) \pm \varphi_R(-x)] \right\}, \quad (15)$$

which satisfy $[\varphi_{\pm}(x), \varphi_{\pm}(y)] = -i\pi \operatorname{sgn}(x - y)$ and $[\varphi_+(x), \varphi_-(y)] = i\pi$. The total Hamiltonian (5)+(6) can then be written as $\mathcal{H}_F + \mathcal{H}_B$, where

$$\mathcal{H}_F = \frac{v}{4\pi} \int_{-\infty}^{\infty} \left(\frac{d\varphi_-}{dx} \right)^2 dx + \frac{\lambda_F}{\pi} \sqrt{\frac{g}{2}} \frac{d\varphi_-(0)}{dx}, \quad (16)$$

$$\mathcal{H}_B = \frac{v}{4\pi} \int_{-\infty}^{\infty} \left(\frac{d\varphi_+}{dx} \right)^2 dx - \frac{\lambda_B}{\pi\alpha} \sin[\sqrt{2g}\varphi_+(0)], \quad (17)$$

and $[\mathcal{H}_F, \mathcal{H}_B] = 0$. In this way the forward and backward scattering processes can be separated into two independent problems. At $g = 1/2$ we fermionize the vertex operator

$$\frac{e^{-i\varphi_+(x)}}{\sqrt{2\pi\alpha}} = \eta\psi_+(x) \quad (18)$$

with a chiral fermion field $\psi_+(x)$ and a Majorana fermion η , which satisfy $\{\psi_+(x), \eta\} = 0$ and $\eta^2 = 1$. The Hamiltonian \mathcal{H}_B can then be refermionized as⁶

$$\mathcal{H}_B = iv \int_{-\infty}^{\infty} \psi_+(x) \frac{d}{dx} \psi_+(x) dx + i \frac{\lambda_B}{\sqrt{2\pi\alpha}} [\eta\psi_+(0) + \psi_+^\dagger(0)\eta], \quad (19)$$

which is a quadratic Hamiltonian and easy to diagonalize. The transmission probability \mathcal{T} is obtained^{6,12} as

$$\mathcal{T}_{g=1/2} = \int_{-\infty}^{\infty} dE \frac{E^2}{E^2 + \Gamma^2} \left(-\frac{d}{dE} \right) \frac{1}{e^{E/T} + 1}, \quad (20)$$

where $\Gamma = \lambda^2/\pi\alpha v_F$.

In the low-energy limit the Hamiltonian (19) is equivalent⁶ to the one obtained by Emery and Kivelson¹³ for the Toulouse limit of the two-channel Kondo model. A “half” of the impurity spin ($S = 1/2$) becomes the Majorana fermion η and the other “half” is decoupled from the Fermi bath, yielding the anomalous residual entropy in the two-channel Kondo problem.

Even away from the special point $g = \frac{1}{2}$, it is possible to solve \mathcal{H}_B exactly with the help of the Bethe ansatz solution to the boundary sine-Gordon model.¹⁴ However, the calculation and resulting formulas are not as simple as in the $g = \frac{1}{2}$ case.

2.2 spinful case

A TLL of electrons in a quantum wire has internal degree(s) of freedom: spin (and “flavor” or band index in nanotubes), and thus it is important to study the quantum impurity problem with spins. Indeed, the spinless

model was generalized to spinful electrons immediately¹⁵ after the spinless case was studied.⁴ Let us briefly review the single-impurity problem for a spinful TLL.

The model we consider is a simple generalization of the spinless case. The field operator of right- and left-going electrons with spin σ are written as

$$\psi_{R\sigma}(x) = \frac{\eta_{\sigma}}{\sqrt{2\pi\alpha}} e^{i\varphi_{R\sigma}(x)}, \quad (21a)$$

$$\psi_{L\sigma}(x) = \frac{\eta_{\sigma}}{\sqrt{2\pi\alpha}} e^{-i\varphi_{L\sigma}(x)}, \quad (21b)$$

where the bosonic fields $\varphi_{R\sigma}$ and $\varphi_{L\sigma}$ satisfy

$$[\varphi_{R\sigma}(x), \varphi_{L\sigma'}(y)] = i\pi\delta_{\sigma,\sigma'} \operatorname{sgn}(x - y), \quad (22a)$$

$$[\varphi_{L\sigma}(x), \varphi_{L\sigma'}(y)] = -i\pi\delta_{\sigma,\sigma'} \operatorname{sgn}(x - y), \quad (22b)$$

$$[\varphi_{R\sigma}(x), \varphi_{L\sigma'}(y)] = i\pi\delta_{\sigma,\sigma'}, \quad (22c)$$

and the Klein factors (Majorana fermions) η_{σ} obeying

$$\{\eta_{\sigma}, \eta_{\sigma'}\} = 2\delta_{\sigma,\sigma'} \quad (23)$$

have been introduced to respect the anticommutation relation between electron fields with antiparallel spins.

Since spin and charge degrees of freedom are separated in a spinful TLL, the low-energy effective theory is given by a sum of two independent Gaussian models. The Hamiltonian of the spinful TLL is given by

$$\mathcal{H}_s = \frac{v_{\rho}}{4\pi} \int_{-\infty}^{\infty} dx \left[\frac{1}{K_{\rho}} \left(\frac{d\phi_{\rho}}{dx} \right)^2 + K_{\rho} \left(\frac{d\theta_{\rho}}{dx} \right)^2 \right] + \frac{v_{\sigma}}{4\pi} \int_{-\infty}^{\infty} dx \left[\frac{1}{K_{\sigma}} \left(\frac{d\phi_{\sigma}}{dx} \right)^2 + K_{\sigma} \left(\frac{d\theta_{\sigma}}{dx} \right)^2 \right], \quad (24)$$

where

$$\phi_{\rho}(x) = \frac{1}{2} [\varphi_{L\uparrow}(x) + \varphi_{L\downarrow}(x) + \varphi_{R\uparrow}(x) + \varphi_{R\downarrow}(x)], \quad (25a)$$

$$\theta_{\rho}(x) = \frac{1}{2} [\varphi_{L\uparrow}(x) + \varphi_{L\downarrow}(x) - \varphi_{R\uparrow}(x) - \varphi_{R\downarrow}(x)], \quad (25b)$$

$$\phi_{\sigma}(x) = \frac{1}{2} [\varphi_{L\uparrow}(x) - \varphi_{L\downarrow}(x) + \varphi_{R\uparrow}(x) - \varphi_{R\downarrow}(x)], \quad (25c)$$

$$\theta_{\sigma}(x) = \frac{1}{2} [\varphi_{L\uparrow}(x) - \varphi_{L\downarrow}(x) - \varphi_{R\uparrow}(x) + \varphi_{R\downarrow}(x)]. \quad (25d)$$

The parameters K_{ρ} and K_{σ} are the TLL parameters controlling charge and spin sectors, respectively. Roughly speaking, they are smaller (larger) than unity when density-density interactions are repulsive (attractive). For noninteracting electrons $K_{\rho} = K_{\sigma} = 1$. For example, in the Hubbard model with on-site repulsion U the parameter K_{ρ} decreases from 1 to $\frac{1}{2}$ as U increases from 0 to infinity. With longer-range interactions K_{ρ} can take a value smaller than $\frac{1}{2}$. It is also important to note that, when the Hamiltonian has SU(2) spin rotation symmetry, K_{σ} is renormalized to 1 in the low-energy limit. The approach to the fixed-point value $K_{\sigma}^* = 1$ is logarithmic and is controlled by the bulk leading irrelevant operator $\cos(2\phi_{\sigma})$, omitted in eq. (24), arising from backward scattering of electrons with antiparallel spins. Although this effect is important for quantitative analysis at finite temperatures, we shall adopt the fixed-point Hamiltonian (24) for our discussion of the impurity model in the

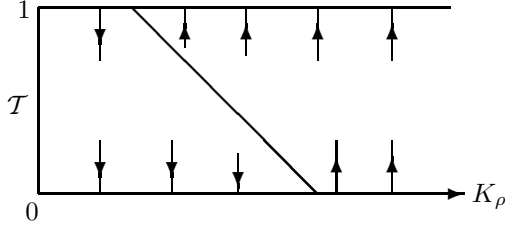


Fig. 2. Schematic flow diagram for the transmission probability at $K_\sigma > 1$. The line of critical points is drawn as a straight line for simplicity.

low-energy limit. We will come back to this issue at the end of this paper.

The backward scattering off the static impurity at $x = 0$ is caused by the impurity Hamiltonian

$$\begin{aligned}\mathcal{H}' &= \lambda_B \sum_{\sigma} \left[\psi_{L\sigma}^\dagger(0) \psi_{R\sigma}(0) + \psi_{R\sigma}^\dagger(0) \psi_{L\sigma}(0) \right] \\ &= -\frac{2\lambda_B}{\pi\alpha} \sin[\phi_\rho(0)] \cos[\phi_\sigma(0)].\end{aligned}\quad (26)$$

Note that there is no spinflip scattering. In the bosonic picture the effect of the impurity backscattering is to pin both charge and spin density fields ϕ_ρ and ϕ_σ . The competition between the kinetic energy (24) of density waves and the pinning potential (26) can be examined through perturbative renormalization group in the same way as in the spinless model. On the one hand, the scaling equation for backscattering λ_B , corresponding to eq. (12), reads

$$\frac{d\lambda_B}{dl} = \frac{1}{2}(2 - K_\rho - K_\sigma)\lambda_B. \quad (27)$$

On the other hand, the scaling equation for fugacity t of single-electron tunneling is generalized from eq. (14) to

$$\frac{dt}{dl} = \frac{1}{2} \left(2 - \frac{1}{K_\rho} - \frac{1}{K_\sigma} \right) t. \quad (28)$$

In the $SU(2)$ symmetric case where $K_\sigma = 1$, the scaling equations (27) and (28) are essentially the same as in the spinless case. Hence the flow diagram (Fig. 1) applies. For repulsive interactions the TLL wire is cut into two semi-infinite wires, while the weak link is healed for attractive interactions.

If we allow K_ρ and K_σ to be free parameters,¹⁵ the flow diagram becomes much richer.^{12, 15–17} For example, when $K_\sigma > 1$ and $2 - K_\sigma < K_\rho < K_\sigma/(2K_\sigma - 1)$, both λ_B and t are renormalized to smaller values, implying that there should be a line (surface) of critical points in the intermediate-coupling regime, where the transmission probability \mathcal{T} takes a nontrivial value. It remains to be understood what kind of boundary conditions should be obeyed by the bosonic fields ϕ_ρ and ϕ_σ at the critical point.

We emphasize that the simplest duality picture (Fig. 1) holds only for the spinless case where there is essentially only one bosonic field ϕ . Once we generalize the model to include more than one bosonic fields as in the spinful case, the problem becomes rather non-trivial. It is a challenging problem to find conformally

invariant boundary conditions, other than the Dirichlet and Neumann boundary conditions we saw in the spinless case. Recent theoretical attempts in this direction can be found in ref. 17–19. A similar problem also arises in the study of quantum and dissipative Josephson junctions.²⁰

3. Orthogonality catastrophe

The orthogonality catastrophe²¹ and Fermi-edge singularities²² have been milestones in the theory of quantum impurities in Fermi liquids. Thus it would be meaningful to briefly review the analogous problem for a TLL.

For simplicity we discuss the orthogonality catastrophe in the spinless case. Let $|0\rangle$ denote the ground state of a clean TLL and $|\lambda\rangle$ the ground state of a TLL with the scattering potential (6) at $x = 0$. The quantity of our interest is the overlap integral $|\langle 0|\lambda\rangle|^2$. The overlap is expected to vanish in the thermodynamic limit as $L^{-\gamma}$, where L is the length of the one-dimensional system. Since the TLL Hamiltonian (5) plus the potential (6) can be separated into two commuting parts \mathcal{H}_F and \mathcal{H}_B , the orthogonality problem can be discussed separately for \mathcal{H}_F and \mathcal{H}_B . Accordingly the exponent γ is written as $\gamma_F + \gamma_B$.

The forward-scattering exponent γ_F can be found easily by a unitary transformation.^{23, 24} Using the commutation relation

$$\frac{\partial}{\partial y} [\varphi_-(x), \varphi_-(y)] = 2\pi i \delta(x - y) \quad (29)$$

we transform the Hamiltonian \mathcal{H}_F as

$$U \mathcal{H}_F U^\dagger = \frac{v}{4\pi} \int dx \left(\frac{d\varphi_-}{dx} \right)^2 + \text{const}, \quad (30)$$

where

$$U = \exp \left[i \frac{\lambda_F}{\pi v} \sqrt{\frac{g}{2}} \varphi_-(0) \right]. \quad (31)$$

The overlap can then be calculated as $|\langle 0|U|0\rangle|^2 \propto L^{-\gamma_F}$ with

$$\gamma_F = 2g \left(\frac{\lambda_F}{2\pi v} \right)^2. \quad (32)$$

The overlap integral in the even (φ_+) sector can be found from the following consideration to take a universal value independent of both mutual interaction strength and impurity potential. In §2.1 we have seen that the renormalization-group flow of the backscattering potential is different for attractive and repulsive interactions. At $g > 1$ the backscattering potential is renormalized to zero in the low-energy limit. This means that there should remain a finite overlap integral even in the thermodynamic limit; $\gamma_B = 0$. At $g < 1$ the backscattering potential grows and eventually the TLL is effectively cut into two decoupled pieces. This means that, roughly speaking, a density wave of even parity changes from $\cos(kx)$ at $\lambda_B = 0$ to $\sin|kx|$ for any $\lambda_B \neq 0$ as $k \rightarrow 0$. The phase shift at $k \rightarrow 0$ is thus $\pi/2$ for any $g < 1$ and $\lambda_B \neq 0$, and we can expect that the exponent γ_B should take a universal value. The exponent can be obtained in various ways.^{11, 25–29} Since the relevant sine potential strongly pins the φ_+ field, one may replace the potential

with a local mass term, $m\varphi_+^2(0)$. Now that the Hamiltonian is quadratic, it is not difficult to compute the overlap.^{25,26} Other approaches include renormalization-group analysis in the weak-interaction limit,²⁷ boundary conformal field theory¹¹ and refermionization^{28,29} at $g = \frac{1}{2}$. All these methods give

$$\gamma_B = \frac{1}{8}. \quad (33)$$

One way to understand this result is use free-fermion picture in the weak-interaction limit.²⁷ It is known²¹ that the orthogonality exponent γ is a sum of odd and even wave functions, $\gamma = (\delta_o/\pi)^2 + (\delta_e/\pi)^2$, where δ_o and δ_e are phase shifts in the wave functions,

$$\psi_o(x) = \sin(kx + \delta_o \text{sgn}x), \quad (34a)$$

$$\psi_e(x) = \cos(kx + \delta_e \text{sgn}x). \quad (34b)$$

The scattering state,

$$\tilde{\psi}(x) = \begin{cases} e^{ikx} + \tilde{r}e^{-ikx}, & x < 0, \\ \tilde{t}e^{ikx}, & x > 0, \end{cases} \quad (35)$$

is related to ψ_o and ψ_e by

$$\tilde{\psi}(x) = ie^{i\delta_o}\psi_o(x) + e^{i\delta_e}\psi_e(x), \quad (36)$$

from which we find the transmission amplitude $\tilde{t} = e^{i\delta_+} \cos \delta_-$ with $\delta_{\pm} = \delta_e \pm \delta_o$. As is obvious from the discussion below eq. (6), the phase shift δ_+ is proportional to λ_F . Since $\mathcal{T} = |\tilde{t}|^2 = \cos^2 \delta_-$, the phase shift δ_- is controlled by λ_B and approaches $\pi/2$ in the low-energy limit (Fig. 1). The exponent γ_B is then given by $(\delta_-/\pi)^2/2 = \frac{1}{8}$.

Another interpretation to eq. (33) is that, in the refermionized Hamiltonian \mathcal{H}_B at $g = \frac{1}{2}$, the impurity (or η) is coupled only to $\psi_+(0) - \psi_+^\dagger(0)$, and the other half degree of freedom $\psi_+ + \psi_+^\dagger$ is decoupled. Hence only half of the fermions ψ_+ have the phase shift $\delta = \pi/2$, leading to the exponent $(\delta/\pi)^2/2 = \frac{1}{8}$.

4. Magnetic impurity

Finally, we turn to the Kondo effect of one-dimensional repulsively interacting electrons. Suppose that there is an impurity spin ($S = \frac{1}{2}$) at the origin. As we have seen in the previous sections, in one dimension we need to distinguish forward and backward scattering by the impurity, unless the impurity spin is at the boundary. We thus consider two kinds of Kondo exchange couplings: forward Kondo scattering J_F and backward Kondo scattering J_B . The Kondo scattering is described by

$$\begin{aligned} \mathcal{H}_J = & \frac{J_F}{2} \mathbf{S} \cdot [\psi_{R\alpha}^\dagger(0) \boldsymbol{\sigma}_{\alpha\beta} \psi_{R\beta}(0) + \psi_{L\alpha}^\dagger(0) \boldsymbol{\sigma}_{\alpha\beta} \psi_{L\beta}(0)] \\ & + \frac{J_B}{2} \mathbf{S} \cdot [\psi_{R\alpha}^\dagger(0) \boldsymbol{\sigma}_{\alpha\beta} \psi_{L\beta}(0) + \psi_{L\alpha}^\dagger(0) \boldsymbol{\sigma}_{\alpha\beta} \psi_{R\beta}(0)], \end{aligned} \quad (37)$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices.

We assume that the bulk TLL is described by the Hamiltonian \mathcal{H}_s (24), where the SU(2) spin rotation symmetry implies $K_\sigma = 1$ and K_ρ is less than 1 (repulsive interaction). The Kondo scattering (37) can be bosonized as in §2.2. The scaling dimension of magnetic

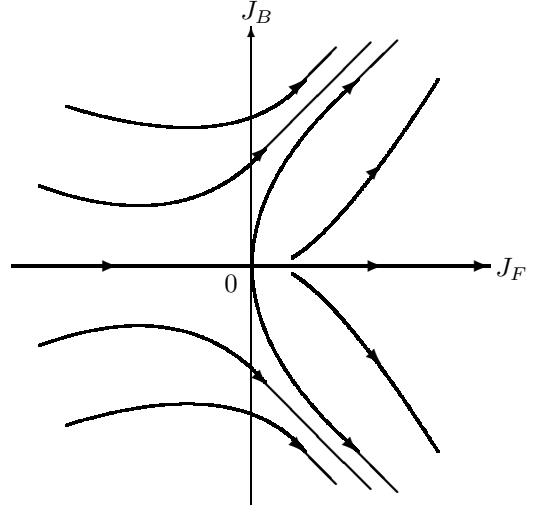


Fig. 3. Schematic flow diagram for the Kondo couplings for $K_\rho < 1$ and $K_\sigma = 1$.

impurity scattering is the same as that of nonmagnetic impurity scattering. Thus, the scaling dimension of J_F is one, whereas J_B has dimension $(1 + K_\rho)/2$. Up to one-loop order, renormalization-group equations can be easily derived using, for example, the poor-man's scaling method.³⁰ The scaling equations read

$$\frac{dJ_F}{dl} = \frac{1}{2\pi v} (J_F^2 + J_B^2), \quad (38a)$$

$$\frac{dJ_B}{dl} = \frac{1}{2} (1 - K_\rho) J_B + \frac{1}{\pi v} J_F J_B. \quad (38b)$$

A schematic flow diagram is shown in Fig. 3.

The flow diagram tells that the trivial fixed point $J_F = J_B = 0$ becomes unstable with infinitesimal J_B , because the backward scattering is relevant for $K_\rho < 1$. This leads to a somewhat surprising conclusion that the Kondo couplings are renormalized towards strong couplings, no matter whether the Kondo couplings are antiferromagnetic or ferromagnetic. This should be contrasted with the standard Fermi liquid case in which the Kondo coupling is renormalized to the strong-coupling regime only if it is antiferromagnetic. Another point to note is that the Kondo temperature depends on the Kondo coupling constant not exponentially but algebraically in TLLs.

The one-loop scaling equations (38a) and (38b) suggest three fixed points besides the trivial one, $(J_F, J_B) = (0, 0)$, mentioned above. These three are $(J_F, J_B) = (+\infty, +\infty)$, $(+\infty, 0)$ and $(+\infty, -\infty)$.

Let us first discuss the case $J_B = 0$. In this case the charge and spin sectors are decoupled in the bosonized Hamiltonian $H_s + H_J$. The impurity spin is interacting with ϕ_s and θ_s only, and the Hamiltonian of the spin sector is equivalent to that of the two-channel Kondo problem.³¹ Here the right- and left-going electrons correspond to two channels. As is well known, the ferromagnetic coupling $J_F > 0$ is renormalized to zero, while the antiferromagnetic coupling flows to the strong-coupling fixed point of the two-channel Kondo model. In the latter

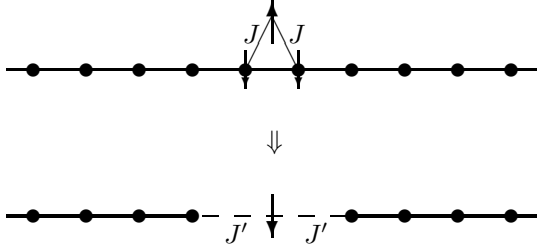


Fig. 4. Lattice model in which an impurity spin is coupled antiferromagnetically to two lattice sites.

case the specific heat and the spin susceptibility acquire anomalous logarithmic contributions due to a local leading irrelevant operator with dimension $\frac{3}{2}$ at the strong-coupling fixed point:

$$\delta C \propto T \ln(T_K/T), \quad \delta \chi \propto \ln(T_K/T). \quad (39)$$

Since the charge excitations have the velocity v_ρ which is different from the spin velocity v_σ , the Wilson ratio is slightly modified⁹ from the standard result $\frac{8}{3}$ to

$$R_W = \frac{4}{3} \left(1 + \frac{v_\sigma}{v_\rho} \right). \quad (40)$$

A simple lattice model realizing this two-channel Kondo physics is depicted in Fig. 4, where an impurity spin $S = \frac{1}{2}$ is coupled to two neighboring sites with equal antiferromagnetic Kondo coupling J . When the electron density is at half filling, one can easily show that the backward scattering vanishes in the continuum limit.^{8,9} Since the charge sector is gapped at half filling, the model is essentially the same as the antiferromagnetic Heisenberg chain with the impurity spin.^{32,33}

We now turn our attention to the generic case $J_F J_B \neq 0$. Since the Kondo couplings always flow towards strong couplings (Fig. 3), it is natural to assume that in the ground state the impurity spin is completely screened as in the standard Kondo effect in Fermi liquids (an exception is the two-channel Kondo case discussed above, which requires the condition $J_B = 0$). We can thus draw schematic strong-coupling pictures of the stable fixed points for (a) antiferromagnetic and (b) ferromagnetic Kondo couplings; see Fig. 5. The ground state consists essentially of two semi-infinite TLLs and a singlet in between them. Since the isolated singlet is expected to have a finite energy gap to excited states, these pictures suggest that the low-energy effective theory is given by two decoupled semi-infinite TLLs plus residual perturbations. Before concluding that these pictures are indeed correct, we have to examine whether the residual interactions are irrelevant at the fixed points. Local operators that could be generated during the renormalization group transformation are the local potential operator at the two ends of the TLLs and the single-electron tunneling between the TLLs. They can be generated by virtual breaking of the singlet due to electron hopping into or out of the singlet. Denoting the end sites of the left and right semi-infinite TLLs by l and r , respectively, we can write these operators as $\psi_\sigma^\dagger(l)\psi_\sigma(l) + \psi_\sigma^\dagger(r)\psi_\sigma(r)$ and

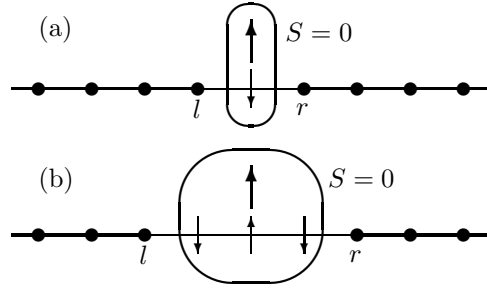


Fig. 5. Schematic pictures of the strong-coupling fixed points for (a) antiferromagnetic and (b) ferromagnetic Kondo couplings.

$\psi_\sigma^\dagger(l)\psi_\sigma(r) + \psi_\sigma^\dagger(r)\psi_\sigma(l)$, where summation over the spin index σ is assumed. The former is exactly marginal and can lead to a shift of the ground state energy. The latter operator is equivalent to the single-electron tunneling discussed in §2.2. It has scaling dimension $(K_\rho^{-1} + 1)/2$ at $K_\sigma = 1$ and is irrelevant in the repulsively interacting electrons ($K_\rho < 1$). We can thus safely conclude that the strong-coupling fixed points are basically two decoupled semi-infinite TLLs, and the impurity spin is completely screened and disappears from the low-energy theory. Treating the tunneling operator as a perturbation, we can compute the impurity contributions to the specific heat and the spin susceptibility at low temperatures. The results are⁸

$$\delta C = c_1 (K_\rho - 1)^2 T^{1/K_\rho - 1} + O(T), \quad (41)$$

$$\delta \chi = c_2 + O(T^2), \quad (42)$$

where c_1 and c_2 are positive constants. When $\frac{1}{2} < K_\rho < 1$, which is the case in the Hubbard model, the low-temperature specific heat has the anomalous power-law contribution as a leading term. This result is confirmed by the boundary conformal field theory analysis^{9,10} as well as by a quantum Monte Carlo calculation.³⁴ The XXZ spin chain with an extra impurity spin also shows a similar temperature dependence.³⁵ If the single-electron tunneling operator is not allowed by symmetry (particle-hole symmetry), then $c_1 = 0$ and the low-temperature behavior is Fermi-liquid like. This is probably what is happening in the solvable toy model studied by Schiller and Ingersent.³⁶

One can also think of the possibility of having a static impurity potential in addition to the magnetic impurity. This is indeed what one would find if one takes the asymmetric Anderson model. Fabrizio and Gogolin³⁷ argued that, if $K_\rho < \frac{1}{2}$ and if the static potential is sufficiently strong, then a situation similar to the one drawn in Fig. 4 occurs and the two-channel Kondo physics is realized. Quantum Monte Carlo calculations³⁴ have obtained results that are consistent with this scenario.

Recent studies have shown that this is not the whole story.^{38,39} As we have noted earlier, the low-energy theory of a spinful TLL is the Gaussian model H_s perturbed by a marginally irrelevant operator $\cos(2\phi_\sigma)$. Even though it is renormalized to zero in the low-energy limit, it should be included in the calculations of finite-temperature quantities like the specific heat and the susceptibility. At the strong-coupling fixed point (Fig. 5)

where the impurity spin is completely screened, the field ϕ_σ obeys a Dirichlet boundary condition at the end sites. As a consequence the first-order perturbation $\langle \cos(2\phi_\sigma) \rangle$ gives a nonvanishing contribution to the free energy. That contribution comes from the region localized near the end sites within the distance of order v_σ/T . This boundary contribution turned out to give leading contribution in δC and $\delta\chi$,

$$\delta C = \frac{1}{2[\ln(T_0/T)]^2}, \quad \delta\chi = \frac{1}{12T\ln(T_0/T)}, \quad (43)$$

where T_0 is the “Kondo” temperature for the bulk marginally irrelevant operator, while the coefficients $\frac{1}{2}$ and $\frac{1}{12}$ are universal. The low-temperature behavior (43) should be easily observed when the impurity potential is sufficiently strong so that the strong-coupling fixed point is already reached at T_0 .

5. Concluding remarks

As we have seen in this paper, the quantum impurity problems in TLLs share many interesting features with the Kondo physics. Some of the theoretical predictions for the simplest case (a static impurity) have been confirmed by tunneling experiments on quantum Hall edge states and carbon nanotubes. Experimental realizations of dynamical impurities are still to be seen in the future. Rapid advances in nanoscience might soon give us such cases.

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